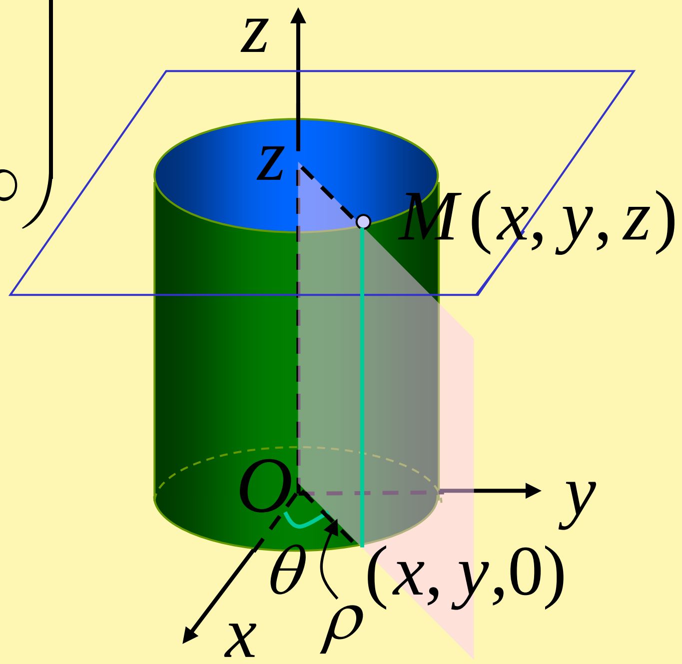
8.3.3、柱面坐标系下的三重积分的计算法

1. 柱面坐标

设 $M(x,y,z)\in \mathbf{R}^3$, 将x,y用极坐标ρ,θ代替,则（ρ,θ,z)就称为点M 的柱坐标. 直角坐标与柱面坐标的关系:

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases} \qquad \begin{pmatrix} 0\le\rho<+\infty \\ 0\le\theta\le 2\pi \\ -\infty<z<+\infty \end{pmatrix}$$



坐标面分别为

$\rho=$ 常数 → 圆柱面

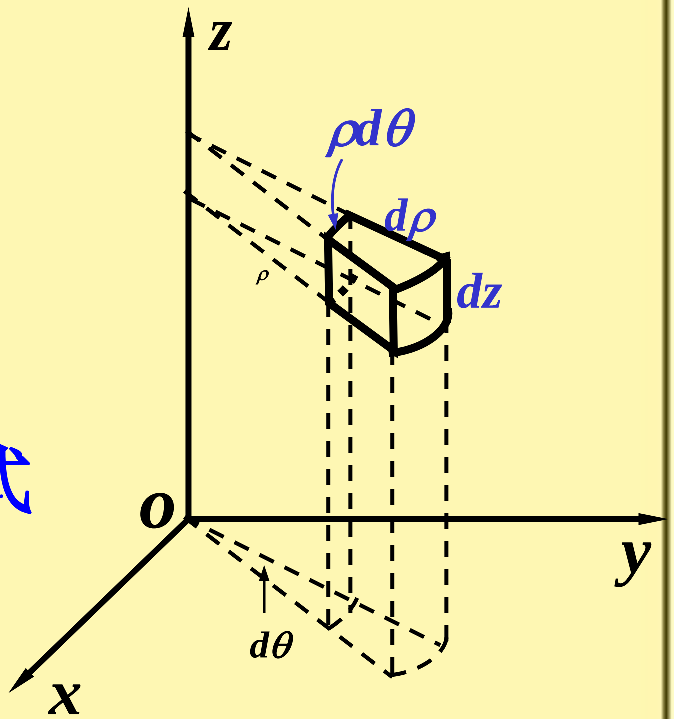
$\theta=$ 常数 → 半平面

$z=$ 常数 → 平面

2．柱面坐标系中的体积元素

如图，柱面坐标系中的体积元素为

$$dv = \rho d\rho d\theta dz,$$



3．柱面坐标系中的三重积分的形式

$$\therefore \iiint_{\Omega} f(x,y,z)dxdydz$$

$$= \iiint_{\Omega} f(\rho\cos\theta, \rho\sin\theta, z)\rho d\rho d\theta dz.$$

4．计算方法：定限方法同直角坐标，把边界化成柱面坐标方程。

例 1 计算 $I=\iiint\limits_{\Omega} z\,dxdydz$，其中 Ω 是球面 $x^2+y^2+z^2=4$ 与抛物面 $x^2+y^2=3z$ 所围的立体.

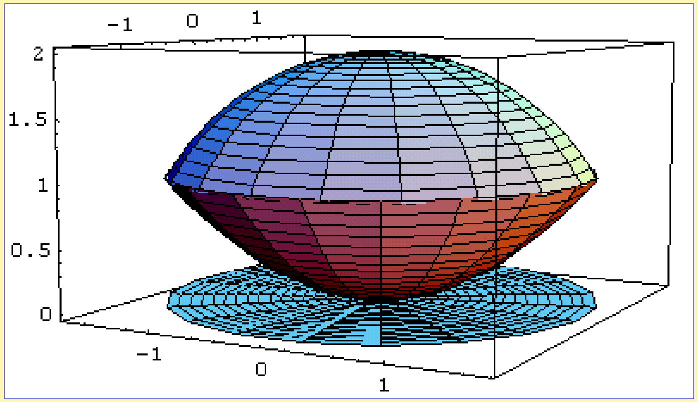
解 $\Omega: \dfrac{x^2+y^2}{3} \le z \le \sqrt{4-x^2-y^2}.(x,y)\in D_{xy}$

投影为: $D_{xy}: x^2+y^2 \le 3$

$$\Omega: \quad \frac{\rho^2}{3} \le z \le \sqrt{4-\rho^2},$$

$$0 \le \rho \le \sqrt{3},$$

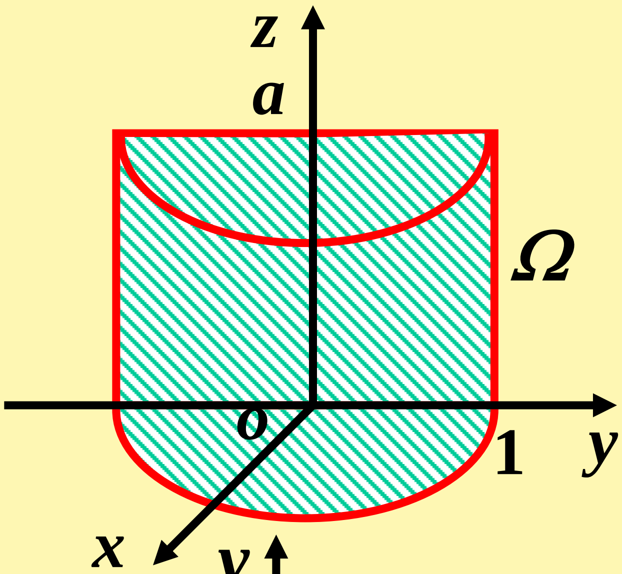
$$0 \le \theta \le 2\pi.$$



$$I=\int_0^{2\pi} d\theta \int_0^{\sqrt{3}} d\rho \int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} \rho \cdot z dz = \frac{13}{4}\pi.$$

例2. 将下列累次积分化为柱面坐标下的累次积分，并计算

$$\int_0^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\int_0^a z\sqrt{x^2+y^2}\,dz$$

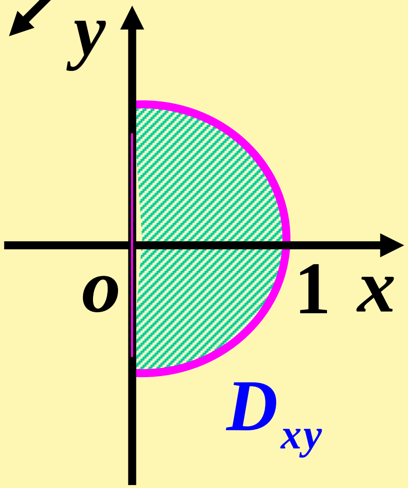


解 $\int_0^1 dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\int_0^a z\sqrt{x^2+y^2}\,dz$

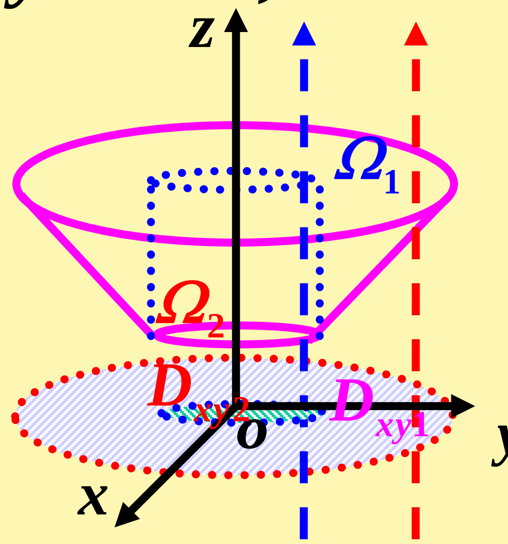
$$=\iiint_{\Omega} z\sqrt{x^2+y^2}\,dv$$

$$=\iint_{D_{xy}} \sqrt{x^2+y^2}\,dxdy\int_0^a zdz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^1 \rho d\rho\int_0^a z\rho dz=\frac{\pi}{6}a^2$$



例3　计算$\iiint_{\Omega}\sqrt{x^2+y^2}dv, \Omega: x^2+y^2\le z^2, 1\le z\le 2$



解(1) $\iiint_{\Omega}\sqrt{x^2+y^2}dv$

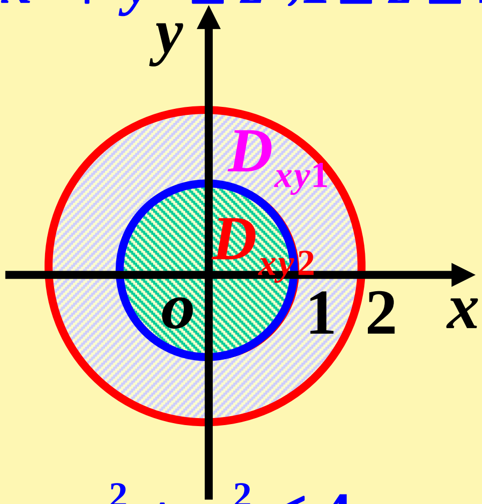
$$=\iiint_{\Omega_1}\sqrt{x^2+y^2}dv+\iiint_{\Omega_2}\sqrt{x^2+y^2}dv$$

$$=\int_0^{2\pi}d\theta\int_1^2\rho\cdot\rho d\rho\int_{\rho}^2 dz$$

$$+\int_0^{2\pi}d\theta\int_0^1\rho\cdot\rho d\rho\int_1^2 dz$$

$$=2\pi[\int_1^2(2\rho^2-\rho^3)d\rho+\frac{1}{3}]=\frac{5\pi}{2}。$$

$\Omega: x^2+y^2\le z^2, 1\le z\le 2$



$D_{xy}: x^2+y^2\le 4$

解法二 用截面法

$$\Omega: \ 1 \le z \le 2, (x, y) \in D_z$$

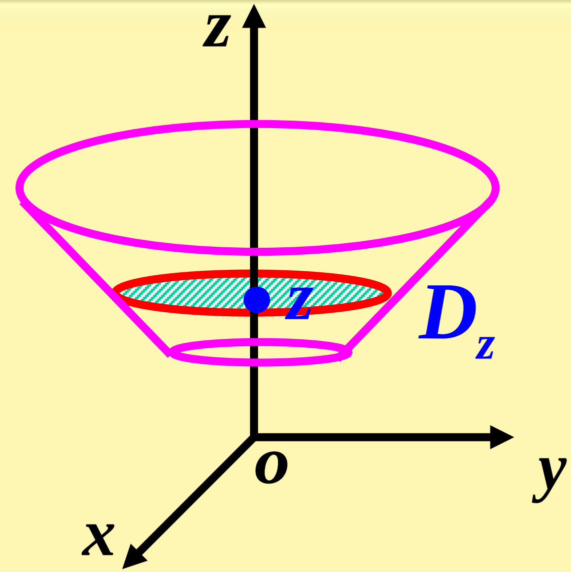
$$\iiint_{\Omega} \sqrt{x^2+y^2}\,dv$$

$$= \int_1^2 dz \iint_{D_z} \sqrt{x^2+y^2}\,dxdy$$

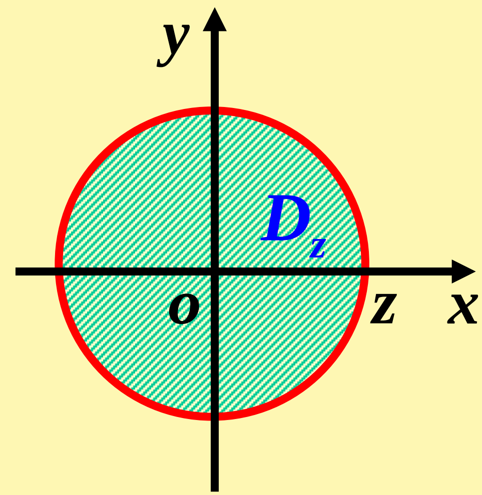
$$= \int_1^2 dz \int_0^{2\pi} d\theta \int_0^z \rho \cdot \rho d\rho$$

$$= 2\pi \cdot \int_1^2 \frac{z^3}{3} dz = \frac{2\pi}{3} \cdot \frac{z^4}{4}\Big|_1^2$$

$$= \frac{5\pi}{2}$$



$$\Omega: x^2+y^2 \le z^2, 1 \le z \le 2$$



$$D_z: x^2+y^2 \le z^2$$

内容小结

1、会选取柱面坐标计算三重积分.

选择柱面坐标计算三重积分依据:

(1) Ω的投影区域或平行截面为圆形域时.

(2) 被积函数形如 $f(x^2+y^2)$、$f(\arctan\frac{y}{x})$, $f(z)$

习题8.3.2

8.3.4、球面坐标系下的三重积分的计算法

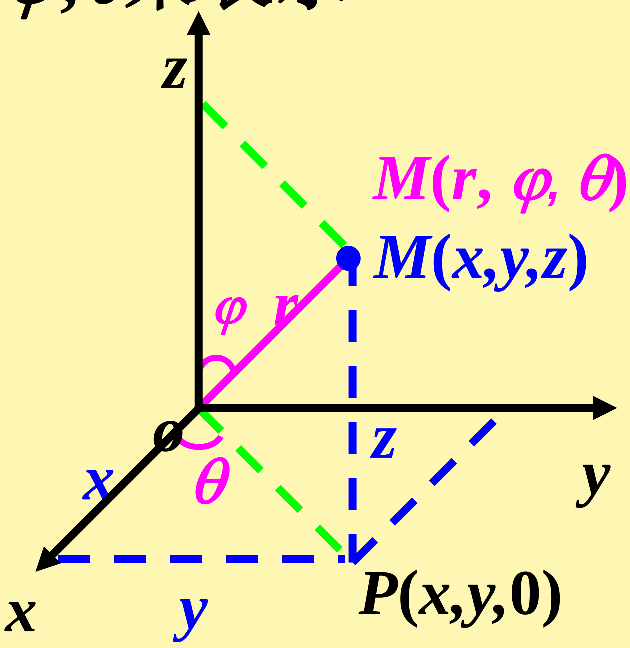
1、球面坐标

设 $M(x,y,z)\in \mathbf{R}^3$，则点M也可用r,φ,θ来表示

r为原点到M间的距离。

φ为有向线段$\overrightarrow{OM}$与z轴正向所夹的角。

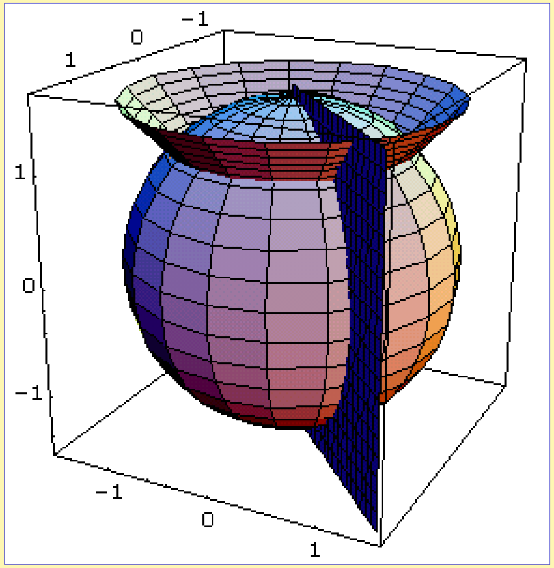
θ为从正z轴来看自x轴按逆时针方向转到有向线段$\overrightarrow{OP}$，这里P是点M在xoy平面上的投影点。



这样三个数r,φ,θ叫做点M的球面坐标。

①球面坐标的变化范围

$$\begin{cases} 0 \le r \le +\infty, \\ 0 \le \varphi \le \pi, \\ 0 \le \theta \le 2\pi \end{cases}$$



②三组坐标面

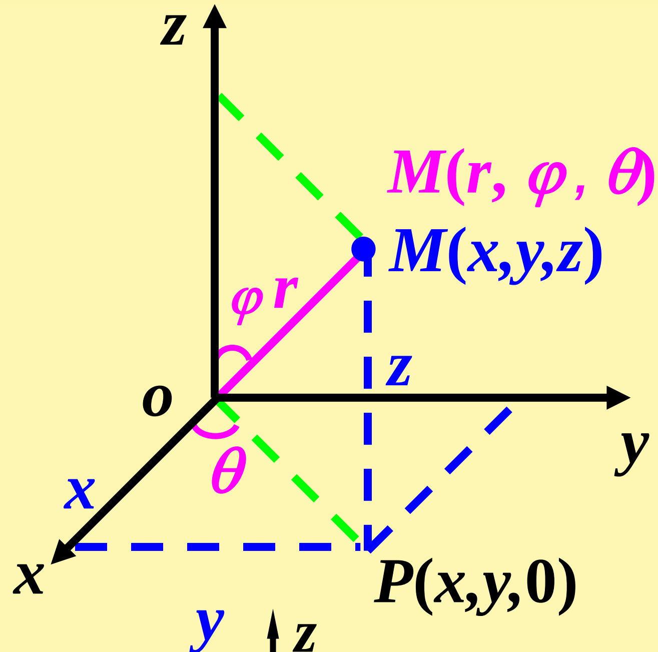
r =常数，即以原点为心的球面。

φ =常数，即以原点为顶点、z轴为轴的圆锥面。

θ =常数，即边z轴的半平面。

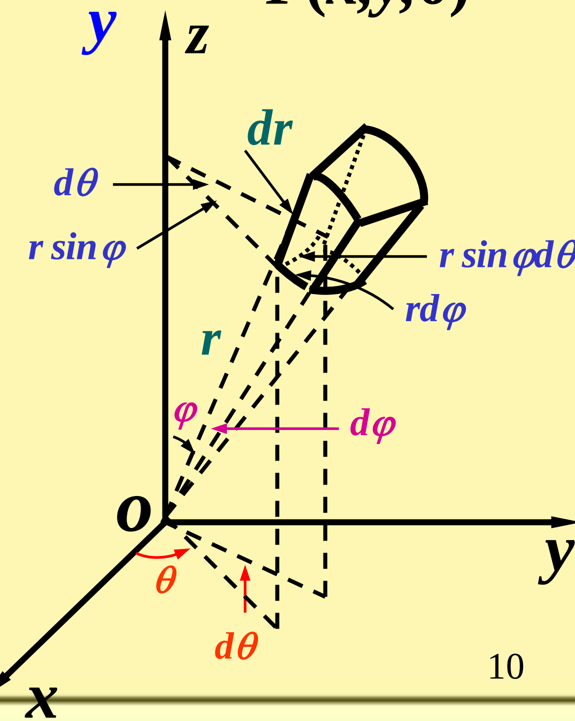
③点*M*的直角坐标与球面坐标的关系为

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$



④球面坐标下的体积元素

$$dv = r^2\sin\varphi drd\varphi d\theta$$



2. 三重积分的球面坐标形式

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{\Omega} F(r,\varphi,\theta)r^2 \sin\varphi dr d\varphi d\theta$$

其中$F(r,\varphi,\theta) = f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)$。

计算三重积分，一般是化为先r，再φ，最后θ的三次积分。

当原点在Ω内时,有

$$0 \le r \le r(\varphi,\theta), 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi,$$

$$\iiint_{\Omega} f(x,y,z)dV = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{r(\varphi,\theta)} F(r,\varphi,\theta) r^2 \sin\varphi dr$$

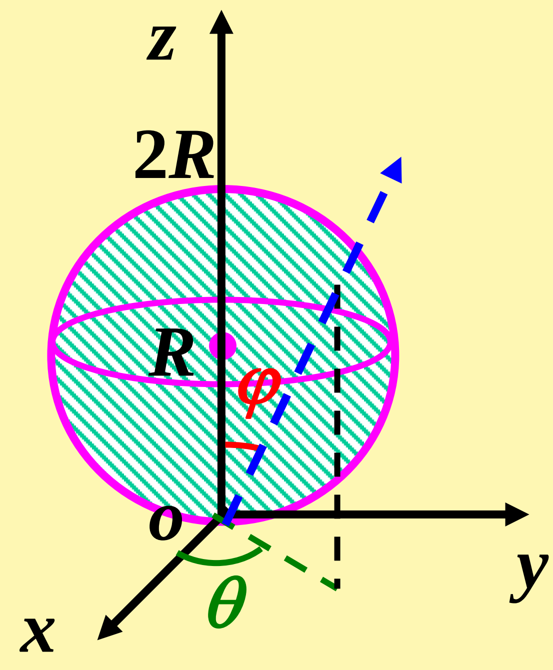
例如，半径为a的球体的体积

$$V=\iiint_{\Omega} dV=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{a}r^2\sin\varphi dr$$

$$=2\pi\cdot 2\cdot\frac{a^3}{3}=\frac{4}{3}\pi a^3。$$

例1 将$\iiint\limits_{\Omega} f(x,y,z)dV$化为球面坐标下的三次积分，其中$\Omega$为：

(1) $\Omega: x^2+y^2+(z-R)^2 \le R^2$



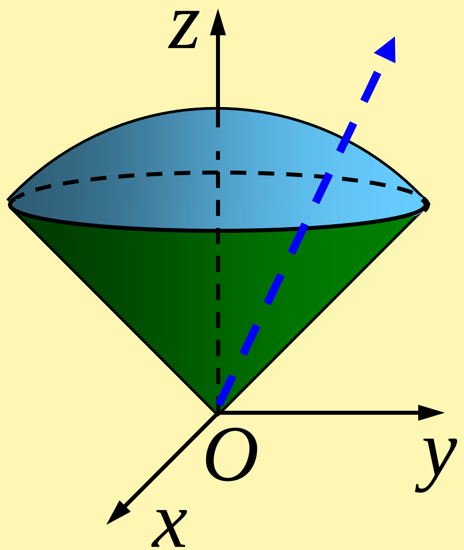
$$\Omega: \begin{cases} 0 \le r \le 2R\cos\varphi \\ 0 \le \varphi \le \dfrac{\pi}{2}, \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\therefore \iiint\limits_{\Omega} f(x,y,z)dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2R\cos\varphi} F(r,\varphi,\theta) r^2 \sin\varphi dr$$

(2) $z=\sqrt{R^2-x^2-y^2}$

与$z=\sqrt{x^2+y^2}$所围

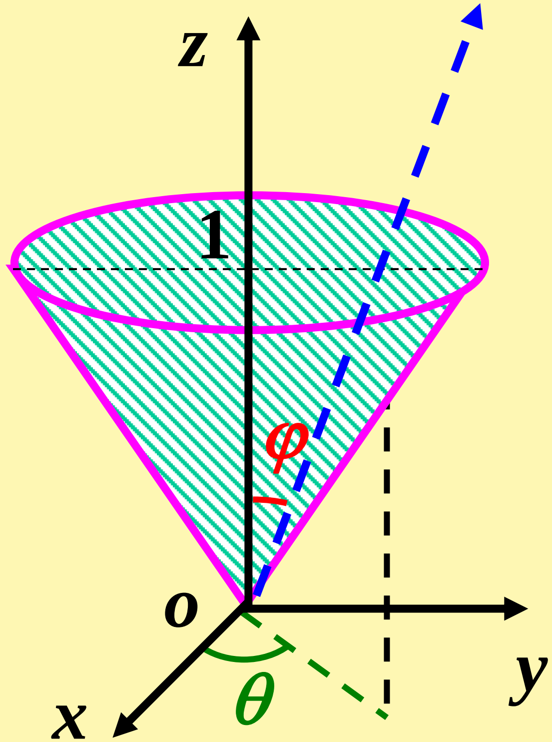


$$\Omega:\begin{cases}0\le r\le R\\ 0\le\varphi\le\dfrac{\pi}{4}\\ 0\le\theta\le 2\pi\end{cases}$$

$$\therefore\iiint\limits_{\Omega}f(x,y,z)dV$$

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{R}F(r,\varphi,\theta)r^2\sin\varphi dr$$

(3) $\Omega:\begin{cases} z \geq k\sqrt{x^2+y^2}\,(k>0) \\ z \leq 1 \end{cases}$



$$\because \Omega: 0 \leq r \leq \frac{1}{\cos\varphi},$$

$$0 \leq \varphi \leq \arctan\frac{1}{k}, 0 \leq \theta \leq 2\pi$$

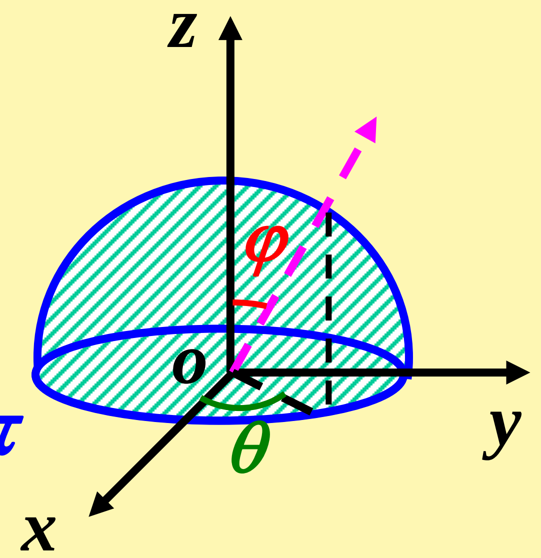
$$\therefore \iiint_{\Omega} f(x,y,z)dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\arctan\frac{1}{k}} d\varphi \int_0^{\frac{1}{\cos\varphi}} F(r,\varphi,\theta) r^2 \sin\varphi dr。$$

例2 将积分先化为球面坐标下的三次积分，并计算

$$(1) I=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}dy\int_{0}^{\sqrt{R^2-x^2-y^2}}(x^2+y^2)dz$$

解 (1) Ω 是以原点为球心,以R为半径的上半球面与xoy面所围成的空间区域。



$$\Omega: 0\le r\le R, 0\le\varphi\le\frac{\pi}{2}, 0\le\theta\le 2\pi$$

$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^R r^2\sin^2\varphi\cdot r^2\sin\varphi dr$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^3\varphi d\varphi\int_0^R r^4 dr=\frac{4}{15}\pi R^5。$$

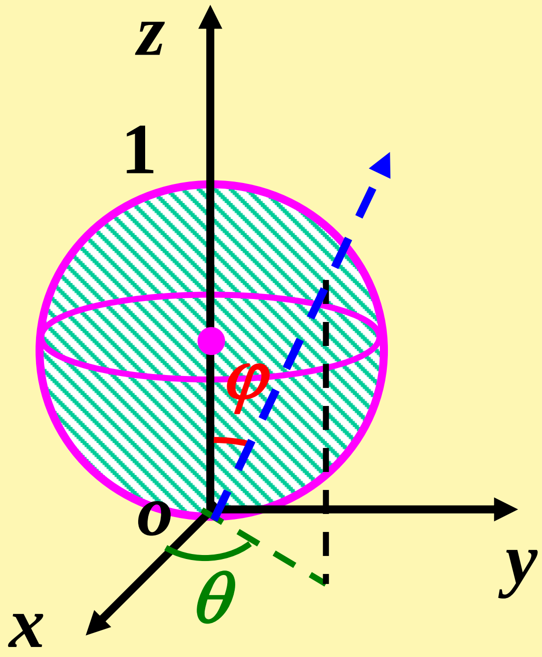
(2) $\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}dv, \Omega: x^2+y^2+z^2 \le z$

解 $\Omega: 0 \le r \le \cos\varphi, 0 \le \varphi \le \dfrac{\pi}{2}, 0 \le \theta \le 2\pi$

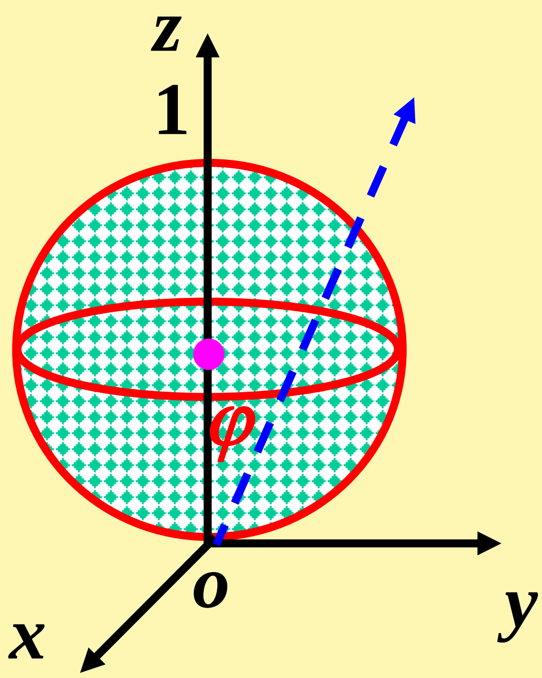
$$\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}dv$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} r \cdot r^2 \sin\varphi dr$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\varphi \frac{\cos^4\varphi}{4} d\varphi = \frac{\pi}{10}$$



(3)求 $\iiint\limits_{\Omega} z^2 dv,\ \Omega: x^2+y^2+z^2 \le 2z$



解法一　用球面坐标系

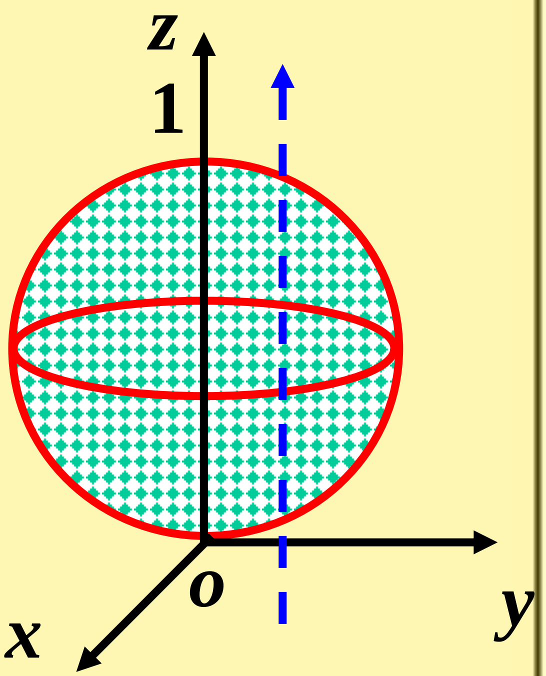
$$\Omega: 0 \le r \le 2\cos\varphi, 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$$

$$I = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \cos^2\varphi \cdot r^2 \sin\varphi dr$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi [\frac{r^5}{5}]_0^{2\cos\varphi} d\varphi$$

$$= \frac{64\pi}{5} \cdot \left(-\frac{\cos^8\varphi}{8} \right) \Bigg|_0^{\frac{\pi}{2}} = \frac{8}{5}\pi。$$

解法二 用柱面坐标系



$$\iiint_{\Omega} z^2 dv = \iint_{D_{xy}} dxdy \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} z^2 dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho d\rho \int_{1-\sqrt{1-\rho^2}}^{1+\sqrt{1-\rho^2}} z^2 dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho \frac{z^3}{3}\Bigg|_{1-\sqrt{1-\rho^2}}^{1+\sqrt{1-\rho^2}} d\rho$$

$$= \frac{2\pi}{3}\int_0^1 \rho[6\sqrt{1-\rho^2} + 2\sqrt{(1-\rho^2)^3} d\rho$$

$$= \frac{4\pi}{3}(-\frac{1}{2})\int_0^1 [3\sqrt{1-\rho^2} + \sqrt{(1-\rho^2)^3}]d(1-\rho^2)$$ 繁！

$$= -\frac{2\pi}{3}\int_1^0 [3\sqrt{u} + \sqrt{u^3}]du = \frac{2\pi}{3}\int_0^1 [3\sqrt{u} + \sqrt{u^3}]du = \frac{8}{5}\pi。$$

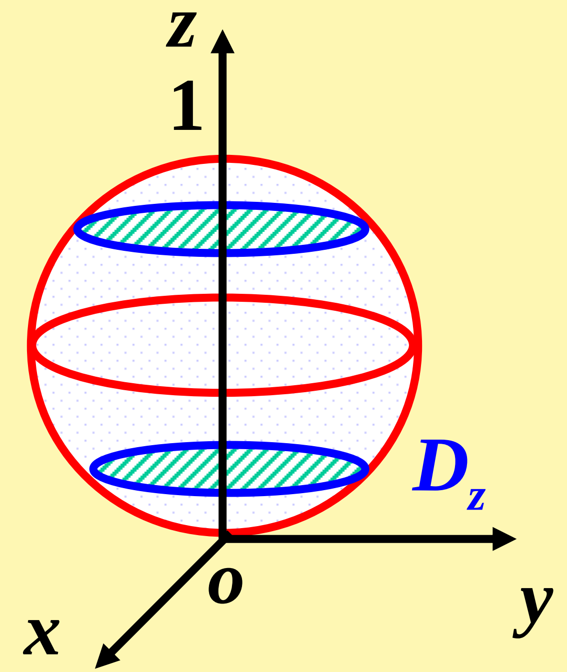
解法三 截面法

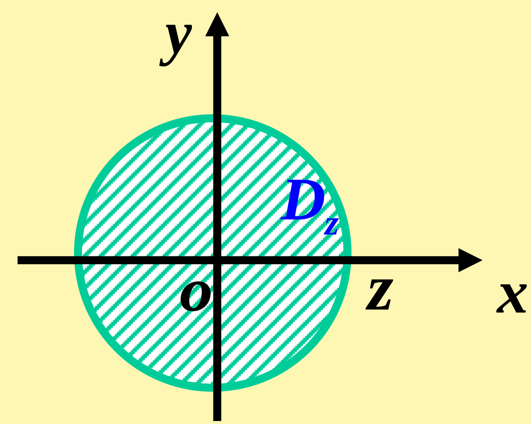
$$\iiint\limits_{\Omega} z^2 dv = \int_0^2 z^2 dz \iint\limits_{D_z} d\sigma$$

$$= \int_0^2 z^2 \pi (2z - z^2) dz$$

$$= \pi \int_0^2 2z^3 - z^4) dz$$

$$= \pi \left(\frac{2}{4} z^4 - \frac{1}{5} z^5 \right) \Big|_0^2 = \frac{8}{5}\pi。$$





$D_z : x^2 + y^2 \le (2z - z^2)$

内容小结

1、会选取球面坐标计算三重积分.

选择球面坐标计算三重积分依据:

（1）被积函数形如 $f(x^2+y^2+z^2)$，$f(x^2+y^2), f(z)$

（2）Ω为球形域，球面与圆锥面所围时.

习题8.3.3