三重积分习题课

基本方法：选择适当的坐标系化三重积分为定积分。

(1).直角坐标系：投影法，截面法。

(2).柱面坐标系　　　(3).球面坐标系

基本技巧：选择适当的坐标系，对称性的应用。

重积分的应用。

空间曲面的面积，空间立体的体积。

物体的质量，质心，转动惯量.

一、 直角坐标下的三重积分的计算方法

1、 投影法；先一后二

$$\Omega : z_1(x,y) \le z \le z_2(x,y), (x,y) \in D_{xy}$$

$$\iiint\limits_{\Omega} f(x,y,z)dv = \iint\limits_{D_{xy}} dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

Ω往另两个坐标面上投影的情况与此类似。

2、 平行截面法；先二后一

$$\Omega = \{(x,y,z) \mid (x,y) \in D_z, c \le z \le d\}$$

则有 $\iiint\limits_{\Omega} f(x,y,z)dv = \int_c^d dz \iint\limits_{D_z} f(x,y,z)dxdy$

特别当$f(x,y,z)$只是 z 的函数：$f(x,y,z)=\varphi(z)$,

$\iiint\limits_{\Omega} f(x,y,z)dv = \int_c^d \varphi(z)A(z)dz$ 其中$A(z)$是D_z的面积

二、 柱面坐标系下计算三重积分

$$\Omega : z_1(x,y) \le z \le z_2(x,y), (x,y) \in D_{xy}$$

即：$z_1(\rho\cos\theta, \rho\sin\theta) \le z \le z_2(\rho\cos\theta, \rho\sin\theta), (\rho,\theta) \in D_{xy}$

$$\iiint_{\Omega} f(x,y,z)dv = \iiint_{\Omega} f(\rho\cos\theta, \rho\sin\theta, z)\rho d\rho d\theta dz$$

$$= \iint_{D_{xy}} \rho d\rho d\theta \int_{z_1(\rho\cos\theta,\rho\sin\theta)}^{z_2(\rho\cos\theta,\rho\sin\theta)} f(\rho\cos\theta, \rho\sin\theta, z)dz$$

（1） Ω的投影区域或平行截面为圆形域时.

(2) 被积函数形如 $f(x^2+y^2)$、$f(\arctan\dfrac{y}{x}), f(z)$

三、球面坐标系下计算三重积分。

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{\Omega} F(r,\varphi,\theta)r^2\sin\varphi drd\varphi d\theta$$

注 Ω以下区域时用球面坐标

1) $\Omega: x^2+y^2+z^2 \le R^2$

$$\iiint_{\Omega} fdxdydz = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^R r^2 fdr$$

2) $\Omega: x^2+y^2+z^2 \le R^2, z \ge 0$

$$\iiint_{\Omega} fdxdydz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^R r^2 fdr$$

3) $\Omega: z=\sqrt{R^2-x^2-y^2}$与$z=\sqrt{x^2+y^2}$所围

$$\iiint_{\Omega} fdxdydz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^R r^2 fdr$$

4) $\Omega: x^2 + y^2 + (z-R)^2 \le R^2$

$$\iiint\limits_{\Omega} f dxdydz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^{2R\cos\varphi} r^2 f dr$$

5) $\Omega: x^2 + y^2 + z^2 \le 2Rz$与$z = \sqrt{x^2 + y^2}$所围

$$\iiint\limits_{\Omega} f dxdydz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^{2R\cos\varphi} r^2 f dr$$

6) $\Omega: 0 < a \le \sqrt{x^2 + y^2 + z^2} \le A$

$$\iiint\limits_{\Omega} f dxdydz = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_a^{A} r^2 f dr$$

有的三重积分可能有多种选择：不同的坐标系、不同的顺序积分等。总结经验，选取简单的方法。

例题分析

一、 填空

1、Ω位于$z=\sqrt{\frac{1}{3}(x^2+y^2)}$之上，球面$x^2+y^2+(z-a)^2=a^2$之上，写出$\iiint\limits_{\Omega} fdxdydz$在球坐标下的累次积分（　　　　）.

$$=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{3}}\sin\varphi d\varphi\int_0^{2a\cos\varphi}r^2fdr$$

2、Ω由$z=\sqrt{x^2+y^2}$，$z=x^2+y^2$所围，写出$\iiint\limits_{\Omega} fdxdydz$在柱坐标下的累次积分（　　　　）. $=\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_{\rho^2}^{\rho}fdz$

3、$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le R^2, z\ge 0\}$,

$\iiint\limits_{\Omega} y\,dxdydz=$（ 0 ）.

4、$\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le a^2, z\ge 0)$，$\Omega_1$为$\Omega$在第一卦限部分，则下式成立的是 C

A $\iiint\limits_{\Omega} x\,dv=4\iiint\limits_{\Omega_1} x\,dv$　　B $\iiint\limits_{\Omega} y\,dv=4\iiint\limits_{\Omega_1} y\,dv$

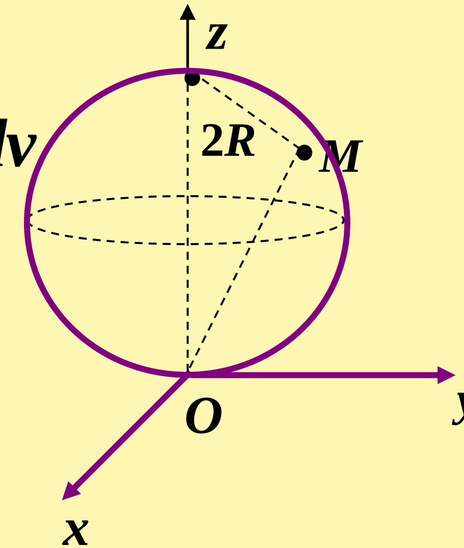
C $\iiint\limits_{\Omega} z\,dv=4\iiint\limits_{\Omega_1} z\,dv$　　D $\iiint\limits_{\Omega} xyz\,dv=4\iiint\limits_{\Omega_1} xyz\,dv$

二、选择适当的坐标系计算

例1　计算$\iiint\limits_{\Omega}(ax+by+cz)dv$ 其中Ω：$x^2+y^2+z^2 \leqslant 2Rz$

解：

由对称性$\iiint\limits_{\Omega} xdv = \iiint\limits_{\Omega} ydv = 0$,只要计算$\iiint\limits_{\Omega} zdv$



解法一：利用球面坐标

$$\Omega: 0 \le r \le 2R\cos\varphi, 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$$

$$I = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^{2R\cos\varphi} r\cos\varphi r^2 \sin\varphi dr$$

$$= 2\pi \int_0^{\pi/2} \cos\varphi \sin\varphi \cdot \frac{r^4}{4}\Big|_0^{2R\cos\varphi} d\varphi$$

$$= 8\pi R^4 \int_0^{\pi/2} \cos^5\varphi \sin\varphi d\varphi = \frac{4\pi}{3} R^4$$

解法二 用柱面坐标计算

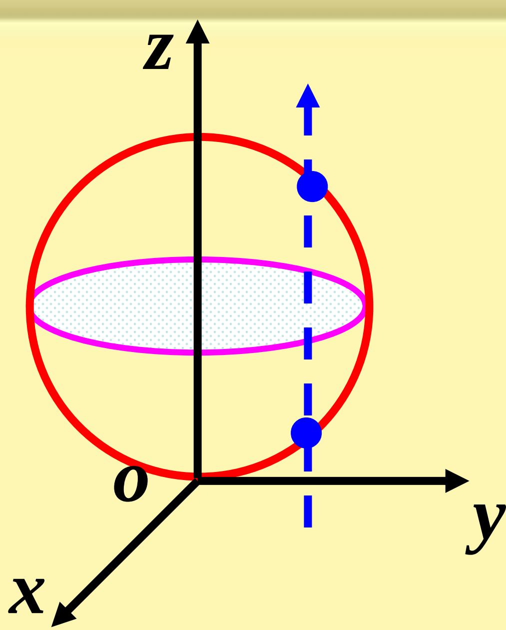
$$\iiint_{\Omega} z\,dv = \iint_{D_{xy}} \left(\int_{R-\sqrt{R^2-x^2-y^2}}^{R+\sqrt{R^2-x^2-y^2}} z\,dz \right) d\sigma$$

$$= \iint_{D_{xy}} \frac{z^2}{2} \Bigg|_{R-\sqrt{R^2-x^2-y^2}}^{R+\sqrt{R^2-x^2-y^2}} d\sigma$$

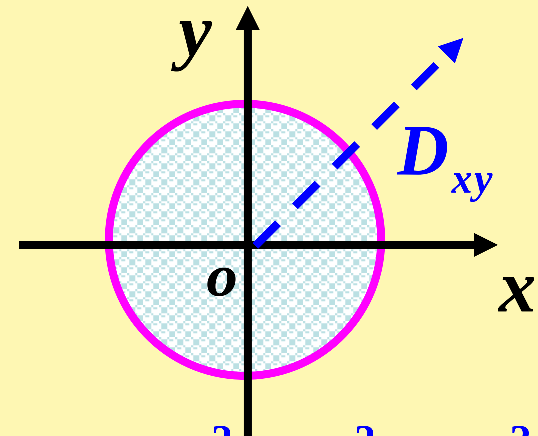
$$= \frac{1}{2} \iint_{D_{xy}} 4R\sqrt{R^2-x^2-y^2}\,d\sigma$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^R 4Rr\sqrt{R^2-r^2}\,dr$$

$$= \frac{4\pi R^4}{3}。$$



$\Omega : x^2 + y^2 + z^2 \le 2Rz$



$D_{xy} : x^2 + y^2 \le R^2$

解法三：利用截面法

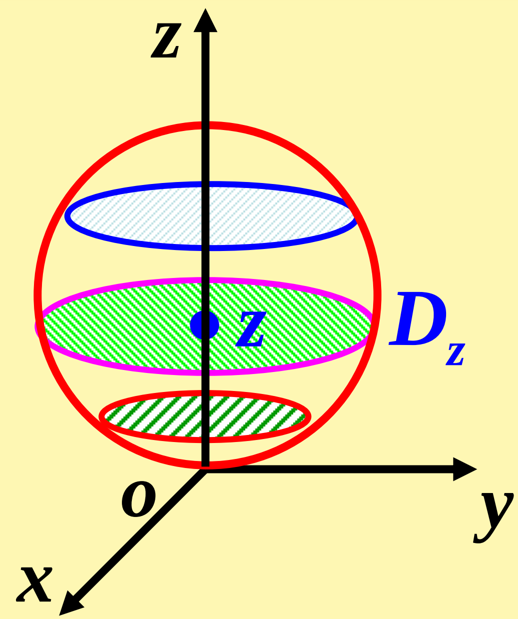
$$\iiint\limits_{\Omega} z\,dv = \int_0^{2R} z\,dz \iint\limits_{D_z} dxdy$$

$$= \int_0^{2R} z\sigma(z)\,dz$$

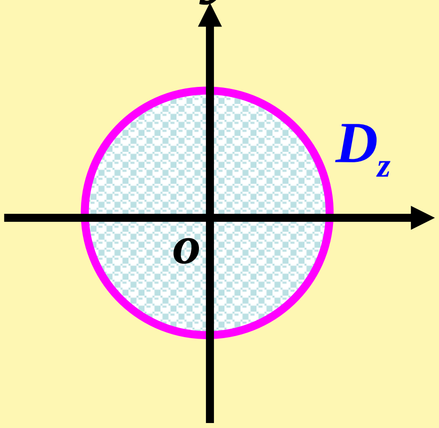
$$= \int_0^{2\pi} \pi(2Rz^2 - z^3)\,dz$$

$$= \pi\left(\frac{2}{3}Rz^3 - \frac{z^4}{4}\right)\Big|_0^{2R}$$

$$= \frac{4}{3}\pi R^4。$$



$$\Omega : x^2 + y^2 + z^2 \le 2Rz$$



$$D_z : x^2 + y^2 \le 2Rz - z^2$$

解法四： 利用质心(形心）概念,球体形心为(0,0,*R*)

$$V=\iiint_{\Omega}dV=\frac{4}{3}\pi R^3$$

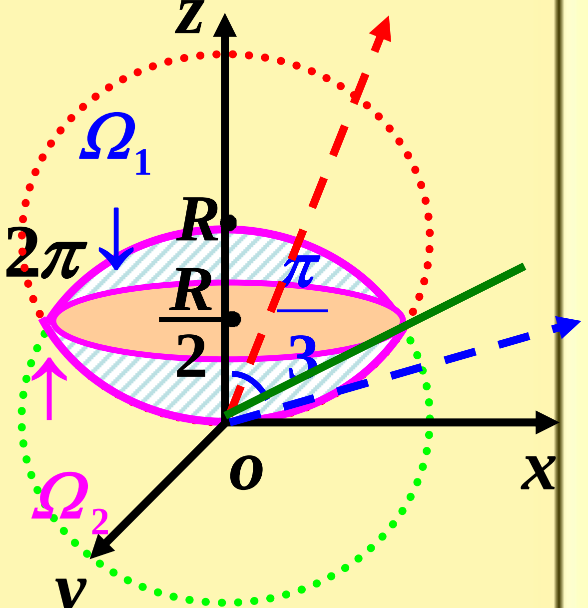
$$\iiint_{\Omega}zdV=\bar{z}V=\frac{4}{3}\pi R^4$$

例3 计算$\iiint\limits_{\Omega} z^2 dV, \Omega: x^2+y^2+z^2 \le a^2, x^2+y^2+z^2 \le 2az$

解法一：用球面坐标

$$\Omega_1: 0 \le r \le a, 0 \le \varphi \le \frac{\pi}{3}, 0 \le \theta \le 2\pi$$

$$\Omega_2: 0 \le r \le 2a\cos\varphi, \frac{\pi}{3} \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$$



$$I = \int_0^{2\pi} d\theta \int_0^{\pi/3} d\varphi \int_0^a r^2\cos^2\varphi \cdot r^2 \sin\varphi dr$$

$$+ \int_0^{2\pi} d\theta \int_{\pi/3}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^4 \cos^2\varphi \sin\varphi dr$$

$$= 2\pi \cdot \left(-\frac{\cos^3\varphi}{3}\right)\Bigg|_0^{\frac{\pi}{3}} \cdot \frac{a^5}{5} + 2\pi \int_{\pi/3}^{\pi/2} \frac{32a^5}{5} \cos^7\varphi \sin\varphi d\varphi$$

$$= \frac{7\pi}{60} a^5 + \frac{\pi}{160} a^5 = \frac{59}{480} a^5$$

解法二：截面法

当$\frac{a}{2} \le z \le a$时， $D_z: x^2 + y^2 \le a^2 - z^2$;

当$0 \le z \le \frac{a}{2}$时， $D_z: x^2 + y^2 \le 2az - z^2$.

$$I = \int_0^{a/2} z^2 dz \iint_{D_z} dxdy + \int_{\frac{a}{2}}^{a} z^2 dz \iint_{D_z} dxdy$$

$$= \int_0^{a/2} \pi(2az - z^2) z^2 dz + \int_{a/2}^{a} \pi(a^2 - z^2) z^2 dz$$

$$= \pi\left(\frac{a}{2} z^4 - \frac{z^5}{5}\right)\Bigg|_0^{\frac{a}{2}} + \pi\left(\frac{a^2}{3} z^3 - \frac{z^5}{5}\right)\Bigg|_{\frac{a}{2}}^{a} = \frac{59}{480}\pi a^5$$

三、作业中存在问题

8.3.2　利用柱坐标计算三重积分

1、 ①$\int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \rho d\rho \int_{\frac{\rho^2}{3}}^{\sqrt{4-\rho^2}} f(\rho^2) dz$

(2)$\int_0^{\frac{\pi}{2}} d\theta \int_0^{1} \rho d\rho \int_0^{1} f(\rho\cos\theta, \rho\sin\theta, z) dz$

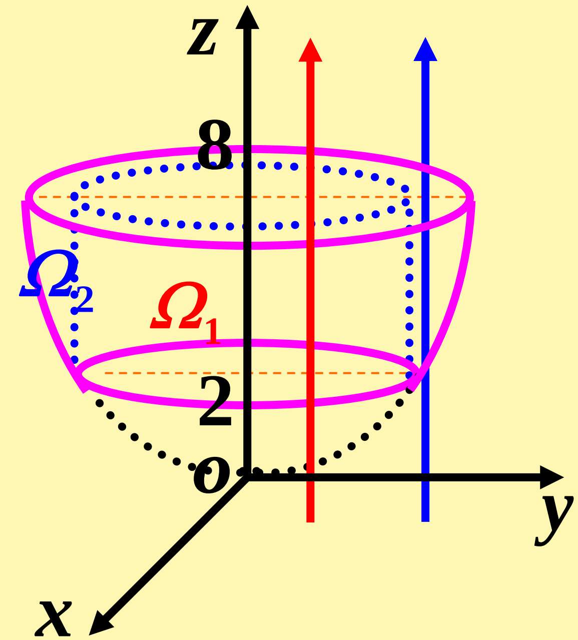
2、 ①$\int_0^{2\pi} d\theta \int_0^{2} \rho d\rho \int_{\frac{5}{2}\rho}^{5} \rho^2 dz = 8\pi$

(2)$\int_0^{2\pi} d\theta \int_0^{1} \rho d\rho \int_{\rho^2}^{\sqrt{2-\rho^2}} z dz = \frac{7}{12}\pi$

（3）法一： 用投影法 分两种情况

$$\Omega_1: 2 \le z \le 8, 0 \le \rho \le 2, 0 \le \theta \le 2\pi$$

$$\Omega_2: \frac{\rho^2}{2} \le z \le 8, 2 \le \rho \le 4, 0 \le \theta \le 2\pi$$



$$\iiint_\Omega (x^2 + y^2) dxdydz$$

$$= \int_0^{2\pi} d\theta \int_0^2 \rho d\rho \int_2^8 \rho^2 dz + \int_0^{2\pi} d\theta \int_2^4 \rho d\rho \int_{\frac{\rho^2}{2}}^8 \rho^2 dz$$

$$= 336\pi$$

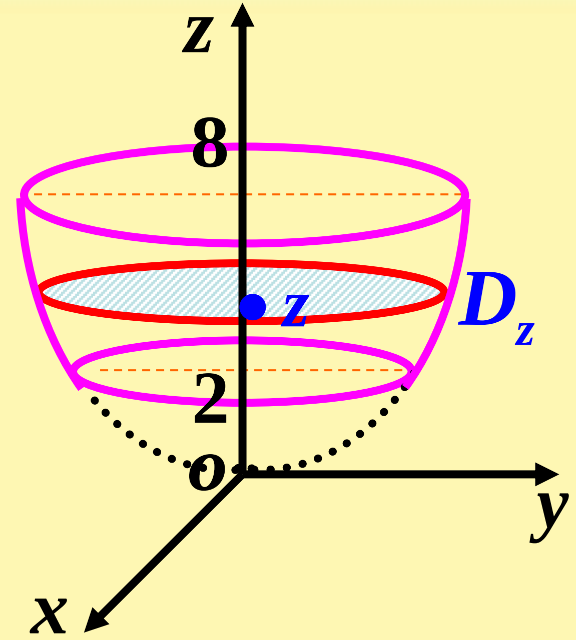
法二：用截面法

$$\iiint_{\Omega}(x^2+y^2)dv$$

$$=\int_2^8 dz \iint_{x^2+y^2\le 2z}(x^2+y^2)dxdy$$

$$=\int_2^8 dz\int_0^{2\pi}d\theta\int_0^{\sqrt{2z}}\rho^2\cdot\rho d\rho$$

$$=336\pi$$



3、$V=\int_0^{2\pi}d\theta\int_0^2\rho d\rho\int_{\rho}^{6-\rho^2}dz=\dfrac{32}{3}\pi$

8.3.3 利用球坐标计算三重积分

1、 ① $\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^{4\cos\varphi} f(r^2\sin^2\varphi + r^3\cos^3\varphi) r^2 dr$

(2) $\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^{1} f(r\sin\varphi\cos\theta, r\sin\theta\sin\varphi, r\cos\varphi) r^2 dr$

2、 ① 利用对称性 $\iiint_{\Omega} x dv = 0,$

原式 $= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi d\varphi \int_0^{1} r\cos\varphi \cdot r^2 dr = \frac{\pi}{8}$

(2) 原式 $= \iiint_{\Omega} (x^2+y^2+z^2) dv + 2\iiint_{\Omega} (xy+yz+xz) dv$

$= \iiint_{\Omega} (x^2+y^2+z^2) dv$ （对称性）

$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{R} r^2 \cdot r^2 dr = \frac{4}{5}\pi R^5$

$$(3)\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_2^4 r^2 \sin^2\varphi \cdot r^2 dr$$

$$= \frac{3968}{15}\pi$$

$$(4)\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^{\cos\varphi} r \cdot r^2 dr = \frac{\pi}{10}$$

四、综合题

(1). 设$f(t)$连续，$f(0)=0$，$f'(0)=1$，求

$$\lim_{t\to 0^+}\frac{1}{\pi t^4}\iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})dV$$

解：$I=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^t f(r)r^2\sin\varphi dr=4\pi\int_0^t r^2 f(r)dr$

原极限 $=\lim_{t\to 0^+}\frac{4}{t^4}\int_0^t r^2 f(r)dr$

$$=\lim_{t\to 0^+}\frac{4t^2 f(t)}{4t^3}=\lim_{t\to 0^+}\frac{f(t)-f(0)}{t}$$

$$=f'(0)=1$$

(2). $f(t)$连续，且恒大于零，

$$F(t)=\frac{\iiint\limits_{\Omega(t)} f(x^2+y^2+z^2)dv}{\iint\limits_{D(t)} f(x^2+y^2)d\sigma},\quad G(t)=\frac{\iint\limits_{D(t)} f(x^2+y^2)d\sigma}{\int_{-t}^{t} f(x^2)dx}$$

其中$\Omega(t)=\{(x,y,z)\,|\,x^2+y^2+z^2\le t^2\}$，

$D(t)=\{(x,y)\,|\,x^2+y^2\le t^2\}$，

(1) 讨论$F(t)$在$(0,+\infty)$上的单调性。

(2) 证明：当$t>0$时，$F(t)>\dfrac{2}{\pi}G(t)$.

证明：$F(t)=\dfrac{\iiint\limits_{\Omega(t)} f(x^2+y^2+z^2)dv}{\iint\limits_{D(t)} f(x^2+y^2)d\sigma}$

$$=\frac{\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^t f(r^2)r^2dr}{\int_0^{2\pi}d\theta\int_0^t f(\rho^2)\rho d\rho}=2\frac{\int_0^t f(r^2)r^2dr}{\int_0^t f(\rho^2)\rho d\rho}$$

$$=2\frac{\int_0^t f(x^2)x^2dx}{\int_0^t f(x^2)xdx}$$

$$F'(t)=2\frac{tf(t^2)[\int_0^t xf(x^2)(t-x)dx]}{[\int_0^t f(x^2)xdx]^2}>0$$

$F(t)$在$(0,+\infty)$上的单调增。

$$G(t)=\frac{\iint\limits_{D(t)} f(x^2+y^2)d\sigma}{\int_{-t}^{t} f(x^2)dx}=\pi\frac{\int_0^t f(r^2)rdr}{\int_0^t f(r^2)dr}$$

要证 $F(t)-\frac{2}{\pi}G(t)=\ 2[\frac{\int_0^t f(r^2)r^2dr}{\int_0^t f(r^2)rdr}-\frac{\int_0^t f(r^2)rdr}{\int_0^t f(r^2)dr}]>0$

令: $\varphi(t)=\int_0^t f(x^2)x^2dx\cdot\int_0^t f(x^2)dx-[\int_0^t f(x^2)xdx]^2$

$$\varphi'(t)=f(t^2)[\int_0^t f(x^2)(x-t)^2dx]>0$$

$\varphi(t)$ 增，$\varphi(0)=0\Rightarrow\varphi(t)>\varphi(0)=0.$

注：利用单调性证明定积分不等式

或把$\varphi(t)=\int_0^t f(x^2)x^2dx\cdot\int_0^t f(x^2)dx-[\int_0^t f(x^2)xdx]^2$

化为二重积分 $D:0\le x\le t, 0\le y\le t$

$$\varphi(t)=\int_0^t f(x^2)x^2dx\cdot\int_0^t f(y^2)dy-\int_0^t f(x^2)xdx\int_0^t f(y^2)ydy$$

$$=\iint_D f(x^2)f(y^2)x^2dxdy-\iint_D f(x^2)f(y^2)xydxdy$$

$$=\iint_D f(x^2)f(y^2)x(x-y)dxdy$$

$$\underline{\underline{x,y\text{互换}}}\iint_D f(x^2)f(y^2)y(y-x)dxdy$$

$$=\frac{1}{2}\iint_D f(x^2)f(y^2)[x-y]^2dxdy>0$$

注：利用二重积分证明定积分不等式