# 第9章　　第二次习题课

## 一、对面积的曲面积分

计算方法: 一代、二换、三投影化为二重积分

设 $\Sigma$: $z=z(x, y)$, $(x, y)\in D_{xy}$

$$\iint\limits_{\Sigma} f(x,y,z)dS=\iint\limits_{D_{xy}} f[x,y,z(x,y)]\sqrt{1+z_x^2(x,y)+z_y^2(x,y)}dxdy$$

注意: 1、利用曲面方程化简被积函数

2、对称性的应用

## 二、对坐标的曲面积分

1、直接计算：“一代二定三投影”化为二重积分计算。

$\sum$：$z=z(x，y)$时，

$$\iint_{\Sigma} R(x,y,z)dxdy = \pm \iint_{D_{xy}} R[x,y,z(x,y)]dxdy$$

上正下负

$\Sigma$ : 由 $x = x(y,z)$ 给出，则有

$$\iint_{\Sigma} P(x,y,z)dydz = \pm \iint_{D_{yz}} P[x(y,z),y,z]dydz$$

前正后负

$\Sigma$ : 由 $y = y(z,x)$ 给出，则有

$$\iint_{\Sigma} Q(x,y,z)dzdx = \pm \iint_{D_{zx}} Q[x,y(z,x),z]dzdx$$

右正左负

2. 利用两类曲面积分之间的关系

$$\iint_{\Sigma}\vec{A}\cdot\vec{dS}=\iint_{\Sigma}(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS$$

3. 投影转换法（法2的改进）

若∑：$z=z(x, y)$，$(x, y)\in D_{xy}$，则

$$\iint_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iint_{D_{xy}}\vec{A}\cdot\vec{N}dxdy.$$

其中 $\vec{A}=\{P,Q,R\},\vec{N}=\pm\{-z_x,-z_y,1\}$ 上正下负

4、高斯公式

$$\oint\!\!\!\oint_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dv$$

注意：利用曲面方程化简被积函数

# 三、高斯公式

## 1. 高斯公式

## 2. 通量与散度

通量（流量）

$$\Phi = \iint_{\Sigma} \vec{A} \cdot d\vec{S} = \iint_{\Sigma} Pdydz + Qdzdx + Rdxdy$$

散度

$$\operatorname{div} A = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

## 四、斯托克斯公式、环流量与旋度

**1. *Stokes*公式**

$$\oint_{\Gamma} Pdx + Qdy + Rdz = \iint_{\Sigma} \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$P$、$Q$、$R$在空间一维单连通区域$G$内一阶偏导连续，$\Sigma$与$\Gamma$符合右手规则。

或

$$\oint_{\Gamma} Pdx + Qdy + Rdz = \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

## 2．环流量与旋度

$$\vec{A}=\{P,Q,R\},$$

$$\operatorname{rot}\vec{A}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)i+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)j+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)k$$

沿有向闭曲线$\Gamma$的曲线积分

$$\oint_{\Gamma} Pdx+Qdy+Rdz$$

叫做向量场$A$沿有向闭曲线 $\Gamma$的环流量。

例1、选择与填空（总习题） $\frac{4\pi R^4}{3}(\alpha^2+\beta^2+\gamma^2)$

1)、$\Sigma$：$x^2+y^2+z^2=a^2$，则 $\oiint\limits_{\Sigma}(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)dS=$ _____

解　利用轮换对称性

$$原式=\frac{1}{3}(\alpha^2+\beta^2+\gamma^2)\iint\limits_{\Sigma}(x^2+y^2+z^2)dS$$

$$=\frac{1}{3}(\alpha^2+\beta^2+\gamma^2)\iint\limits_{\Sigma}R^2dS$$

$$=\frac{4\pi R^4}{3}(\alpha^2+\beta^2+\gamma^2)$$

2)、$\Sigma$：$yoz$平面上的圆域$y^2+z^2\le 1$,

则$\iint\limits_{\Sigma}(x^2+y^2+z^2)dS=$ ___$\frac{\pi}{2}$___

解 $\Sigma: z=0$

原式$=\iint\limits_{D_{yz}}(y^2+z^2)dydz=\int_0^{2\pi}d\theta\int_0^1\rho^3 d\rho=\frac{\pi}{2}$

3）、设$r=\sqrt{x^2+y^2+z^2}$， 则

$\text{div}(\text{grad}\, r)=$ ___$2/r$___ ; $\text{rot}(\text{grad}\, r)=$ ___$\vec{0}$___ .

解: $\text{grad}\, r=\{\frac{x}{r},\frac{y}{r},\frac{z}{r}\}$

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right)=\frac{r-x\cdot\frac{x}{r}}{r^2}=\frac{r^2-x^2}{r^3}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r}\right)=\frac{r^2-y^2}{r^3} \qquad \frac{\partial}{\partial z}\left(\frac{z}{r}\right)=\frac{r^2-z^2}{r^3}$$

三式相加即得 $\operatorname{div}(\operatorname{grad} r)=\dfrac{2}{r}$

$$\operatorname{rot}(\operatorname{grad} r)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{x}{r} & \dfrac{y}{r} & \dfrac{z}{r}\end{vmatrix} =\{0,0,0\}$$

$$=\left[\frac{\partial(\frac{z}{r})}{\partial y}-\frac{\partial(\frac{y}{r})}{\partial z}\right]\vec{i}+\left[\frac{\partial(\frac{x}{r})}{\partial z}-\frac{\partial(\frac{z}{r})}{\partial x}\right]\vec{j}+\left[\frac{\partial(\frac{y}{r})}{\partial x}-\frac{\partial(\frac{x}{r})}{\partial y}\right]\vec{k}.$$

$$=\left[\frac{-yz}{r^3}-\frac{-yz}{r^3}\right]\vec{i}+\left[\frac{-xz}{r^3}-\frac{-xz}{r^3}\right]\vec{j}+\left[\frac{-xy}{r^3}-\frac{-xy}{r^3}\right]\vec{k}.$$

例2 $\iint\limits_{\Sigma}(x^2+y^2+z^2)dS \quad \Sigma \quad x^2+y^2+z^2=2ax$

解 记$\Sigma_1 \quad x-a=\sqrt{a^2-z^2-y^2}$,

$$\Sigma_2 \quad x-a=-\sqrt{a^2-z^2-y^2}$$

则 $\dfrac{\partial x}{\partial y}=\dfrac{-y}{\sqrt{a^2-z^2-y^2}}, \quad \dfrac{\partial x}{\partial z}=\dfrac{-z}{\sqrt{a^2-z^2-y^2}}$

$$ds=\frac{a}{\sqrt{a^2-z^2-y^2}}dydz \quad D_{yz} \quad y^2+z^2\le a^2$$

$$\iint\limits_{\Sigma}(x^2+y^2+z^2)dS=\iint\limits_{\Sigma}2axdS=\iint\limits_{\Sigma_1}2axdS+\iint\limits_{\Sigma_2}2axdS$$

$$=\iint\limits_{\Sigma_1}2a(a+\sqrt{a^2-y^2-z^2})dS$$

$$+\iint\limits_{\Sigma_2}2a(a-\sqrt{a^2-y^2-z^2})dS$$

$$= \iint_{\Sigma_1} 2a^2 dS + \iint_{\Sigma_2} 2a^2 dS + \iint_{\Sigma_2} 2a\sqrt{a^2 - y^2 - z^2})dS$$

$$- \iint_{\Sigma_2} 2a\sqrt{a^2 - y^2 - z^2})dS$$

$$= 8\pi a^4 + 2a\iint_{D_{yz}} adydz - 2a\iint_{D_{yz}} adydz = 8\pi a^4$$

或 $$\iint_{\Sigma} (x^2 + y^2 + z^2)dS = \iint_{\Sigma} 2axdS = 2a\bar{x} \cdot S$$

$$= 2a \cdot a \cdot 4\pi a^2 = 8\pi a^4$$

注：此题可改成$\iint_{\Sigma} (x^2 + y^2 + z^2)dS \quad \Sigma : x^2 + y^2 + z^2 = 2az$

例3 计算$\iint\limits_{\Sigma}(y^2-z)dydz+(z^2-x)dzdx+(x^2-y)dxdy$

$\Sigma$为$z=\sqrt{x^2+y^2}\,(0\le z\le h)$的外侧

解：方法1：直接计算

$$\iint\limits_{\Sigma}(y^2-z)dydz=\iint\limits_{\Sigma_1}+\iint\limits_{\Sigma_2}$$

$\Sigma_1$是$\Sigma$的前半部分，
$\Sigma_2$是$\Sigma$的后半部分

$$=\iint\limits_{D_{yz}}(y^2-z)dydz-\iint\limits_{D_{yz}}(y^2-z)dydz=0$$

类似地 $\iint\limits_{\Sigma}(z^2-x)dzdx=0$

而 $\iint\limits_{\Sigma}(x^2-y)dxdy=-\iint\limits_{D_{xy}}(x^2-y)dxdy$ $\quad D_{xy}:x^2+y^2\le h^2$

$$I=-\iint\limits_{D_{xy}}(x^2-y)dxdy=-\frac{1}{2}\iint\limits_{D_{xy}}(x^2+y^2)dxdy$$

$$=-\frac{1}{2}\int_0^{2\pi}d\theta\int_0^h\rho^3d\rho=-\frac{\pi}{4}h^4$$

方法2 投影转换 $\vec{N}=\left\{\dfrac{x}{\sqrt{x^2+y^2}},\dfrac{y}{\sqrt{x^2+y^2}},-1\right\}$

$$\vec{A}=\{y^2-z,z^2-x,x^2-y\}$$

$$I=\iint_{D_{xy}}\vec{A}\cdot\vec{N}dxdy \qquad D_{xy}:x^2+y^2\le h^2$$

$$=\iint_{D_{xy}}\left[\frac{x}{\sqrt{x^2+y^2}}(y^2-z)+\frac{y}{\sqrt{x^2+y^2}}(z^2-x)-(x^2-y)\right]dxdy$$

$$=\iint_{D_{xy}}\left[\left(\frac{xy^2}{\sqrt{x^2+y^2}}-x+y\sqrt{x^2+y^2}-\frac{xy}{\sqrt{x^2+y^2}}\right)-(x^2-y)\right]dxdy$$

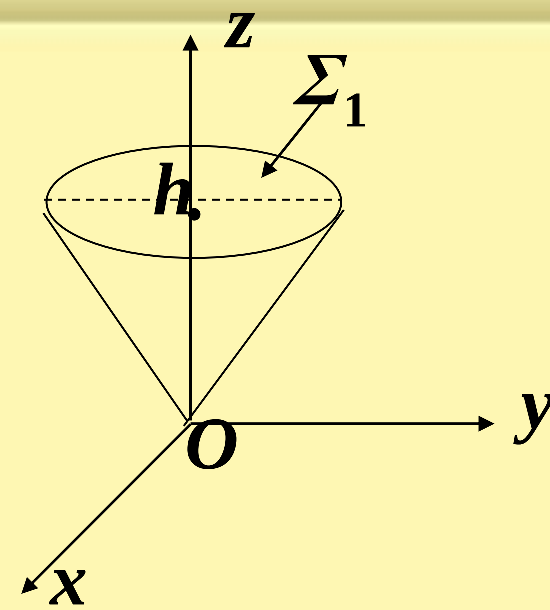
$$=0-\iint_{D_{xy}}x^2dxdy+0=-\frac{\pi}{4}h^4$$

## 方法3　高斯公式

$\Sigma_1 \quad x^2+y^2\le h^2, z=h$ 取上侧，则

$$\iint_{\Sigma} = \oiint_{\Sigma+\Sigma_1} - \oiint_{\Sigma_1} = \iiint_{\Omega} 0dv - \iint_{\Sigma_1}(x^2-y)dxdy$$

$$= -\iint_{D_{xy}}(x^2-y)dxdy = -\frac{\pi}{4}h^4$$




注：计算 $\iint_{\Sigma} Pdydz+Qdzdx+Rdxdy$

若$\Sigma$为平面，一般化为第一类或投影转换，

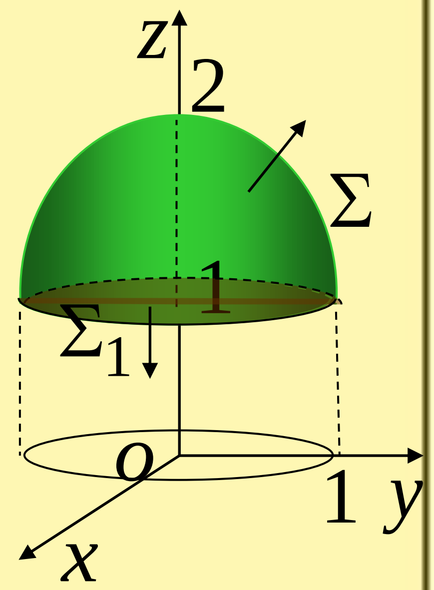
若$\Sigma$为曲面，一般补面用高斯公式

**例4** 设Σ 为曲面$z = 2 - x^2 - y^2$，$1 \le z \le 2$ 取上侧，求

$$I = \iint_{\Sigma} (x^3 z + x)dydz - x^2 yzdzdx - x^2 z^2 dxdy.$$

**解:** 作取下侧的辅助面

$$\Sigma_1 : z = 1 \quad (x, y) \in D_{xy} : x^2 + y^2 \le 1$$




$$I = \oiint_{\Sigma+\Sigma_1} - \iint_{\Sigma_1}$$

用柱坐标

用极坐标

$$= \iiint_{\Omega} dxdydz - (-1)\iint_{D_{xy}} (-x^2)dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho d\rho \int_1^{2-\rho^2} dz - \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 \rho^3 d\rho$$

$$= \frac{13\pi}{12}$$

例5 计算$\iint\limits_{\Sigma}(x^3+az^2)dydz+(y^3+ax^2)dzdx+(z^3+ay^2)dxdy$

$\Sigma$是上半球面$z=\sqrt{a^2-x^2-y^2}$的外侧$(a>0)$.

解：补面$\Sigma_1: z=0$, 取下侧

$$原式=3\iiint\limits_{\Omega}(x^2+y^2+z^2)\,dxdydz-\iint\limits_{\Sigma_1}$$

$$=3\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi d\varphi\int_0^R r^2\cdot r^2dr+a\iint\limits_D y^2dxdy$$

$$=\frac{6}{5}\pi a^5+\frac{a}{2}\iint\limits_D(x^2+y^2)dxdy$$

$$=\frac{6}{5}\pi a^5+\frac{a}{2}\int_0^{2\pi}d\theta\int_0^a\rho^2\rho d\rho=\frac{29}{20}\pi a^5$$

例6.计算$\iint_{\Sigma}\frac{\cos(\vec{r},\vec{n})}{|\vec{r}|^2}dS$, $\Sigma$为一封闭曲面$\vec{n}$为$\Sigma$上点$(x,y,z)$处的单位外法向量$\vec{r}=\{x,y,z\}$.

解：原式$=\iint_{\Sigma}\frac{\vec{r}\cdot\vec{n}}{|\vec{r}|^3}dS=\iint_{\Sigma}\frac{x\cos\alpha+y\cos\beta+z\cos\gamma}{\sqrt{(x^2+y^2+z^2)^3}}dS$

$$=\iint_{\Sigma}\frac{xdydz+ydzdx+zdxdy}{\sqrt{(x^2+y^2+z^2)^3}}$$

$$\frac{\partial P}{\partial x}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3}$$

$$=\frac{y^2+z^2-2x^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

由对称性知

$$\frac{\partial Q}{\partial y}=\frac{z^2+x^2-2y^2}{\sqrt{(x^2+y^2+z^2)^5}} \quad \frac{\partial R}{\partial z}=\frac{x^2+y^2-2z^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

所以除原点外处处有 $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$

(1)：$\Sigma$不包围原点， $\iint_{\Sigma}\frac{\cos(\vec{r},\vec{n})}{|\vec{r}|^2}dS=0$

(2)：$\Sigma$包围原点，

$$\text{原式}=\iint_{\Sigma_1}\frac{xdydz+ydzdx+zdxdy}{\sqrt{(x^2+y^2+z^2)^3}},\Sigma_1: x^2+y^2+z^2=\varepsilon^2$$

$$=\frac{1}{\varepsilon^3}\iint_{\Sigma_1}xdydz+ydzdx+zdxdy=\frac{1}{\varepsilon^3}\iiint_{\Omega}3dv=4\pi$$

例7.流速$\vec{v}(x,y,z)=\{x^3,y^2,z^4\}$的液体流过由曲面$z=4-(x^2+y^2)$和$z=1-\frac{1}{4}(x^2+y^2)$所围的立体。今用平行于$xoz$面的平面截此立体，问沿$y$轴正方哪个截面的流量 最大?

解：$\Sigma_{y_0}$ : 平面$y=y_0$上由曲线$z=4-(x^2+y_0{}^2)$和$z=1-\frac{1}{4}(x^2+y_0{}^2)$所围$(|y_0|\le 2)$, 取右侧

$$\Phi(y_0)=\iint_{\Sigma_{y_0}} x^3dydz+y^2dzdx+z^4dxdy=\iint_{\Sigma_{y_0}} y^2dzdx$$

$$=\int_{-\sqrt{4-y_0{}^2}}^{\sqrt{4-y_0{}^2}}dx\int_{1-\frac{1}{4}y_0{}^2-\frac{1}{4}x^2}^{4-y_0{}^2-x^2} y_0{}^2dz$$

$$=\int_{-\sqrt{4-y_0^2}}^{\sqrt{4-y_0^2}}dx\int_{1-\frac{1}{4}y_0^2-\frac{1}{4}x^2}^{4-y_0^2-x^2}y_0^2dz$$

$$=\int_{-\sqrt{4-y_0^2}}^{\sqrt{4-y_0^2}}\frac{3}{4}(4-y_0^2-x^2)y_0^2dx=(4-y_0^2)^{\frac{3}{2}}y_0^2$$

$$\Phi'(y_0)=(4-y_0^2)^{\frac{1}{2}}y_0(8-5y_0^2)=0$$

$$y_0=\pm\sqrt{\frac{8}{5}}$$

比较$\Phi(0)=0,\Phi(\pm2)=0,\Phi(\pm\sqrt{\frac{8}{5}})=\frac{192}{125}\sqrt{15}$

通过截面$y_0=\pm\sqrt{\frac{8}{5}}$的流量最大

例8.在变力$\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$的作用下，质点沿直线运动到椭球面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$上第一卦限的点$(\xi,\eta,\varsigma)$，问$\xi,\eta,\varsigma$取何值时，力所做的功最大？并求最大值。

解：空间直线$\overline{OM}$的参数方程为：$x=\xi t, y=\eta t, z=\varsigma t,$

$0\le t\le 1$

$$W=\int_{\overline{OM}}\vec{F}\cdot\vec{ds}=\int_{\overline{OM}}yzdx+zxdy+xydz=\int_0^1 3\xi\eta\varsigma t^2dt=\xi\eta\varsigma$$

下求$W=\xi\eta\varsigma$在$\dfrac{\xi^2}{a^2}+\dfrac{\eta^2}{b^2}+\dfrac{\varsigma^2}{c^2}=1(\xi,\eta,\varsigma>0)$

条件下的最大值

求得$\xi=\dfrac{a}{\sqrt{3}},\eta=\dfrac{b}{\sqrt{3}},\varsigma=\dfrac{c}{\sqrt{3}}$　　$W_{最大值}=\dfrac{\sqrt{3}}{9}abc$